

Solutions to Assignment 3, MAS 231

1. $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$

$$\vec{b} \times \vec{c} = (b_2 c_3 - b_3 c_2, b_3 c_1 - b_1 c_3, b_1 c_2 - b_2 c_1)$$

$$\begin{aligned} \vec{a} \times (\vec{b} \times \vec{c}) = & (a_2(b_1 c_2 - b_2 c_1) - a_3(b_3 c_1 - b_1 c_3), \\ & a_3(b_2 c_3 - b_3 c_2) - a_1(b_1 c_2 - b_2 c_1), \\ & a_1(b_3 c_1 - b_1 c_3) - a_2(b_2 c_3 - b_3 c_2)) \quad (*) \end{aligned}$$

$$\begin{aligned} (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} = & (a_1 c_1 + a_2 c_2 + a_3 c_3) \cdot (b_1, b_2, b_3) \\ & - (a_1 b_1 + a_2 b_2 + a_3 b_3) (c_1, c_2, c_3) \end{aligned}$$

$$\begin{aligned} = & \left((a_1 c_1 + a_2 c_2 + a_3 c_3) b_1 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_1, \right. \\ & (a_1 c_1 + a_2 c_2 + a_3 c_3) b_2 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_2, \\ & \left. (a_1 c_1 + a_2 c_2 + a_3 c_3) b_3 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_3 \right) \end{aligned}$$

$$\begin{aligned} = & (a_2 c_2 b_1 + a_3 c_3 b_1 - a_1 b_1 c_1 - a_3 b_3 c_1, \\ & a_1 c_1 b_2 + a_3 c_3 b_2 - a_1 b_1 c_2 - a_3 b_3 c_2, \\ & a_1 c_1 b_3 + a_2 c_2 b_3 - a_1 b_1 c_3 - a_2 b_2 c_3) \end{aligned}$$

$$\begin{aligned} = & (a_2(b_1 c_2 - b_2 c_1) - a_3(b_3 c_1 - b_1 c_3), \\ & a_3(b_2 c_3 - b_3 c_2) - a_1(b_1 c_2 - b_2 c_1), \\ & a_1(b_3 c_1 - b_1 c_3) - a_2(b_2 c_3 - b_3 c_2)) \quad (***) \end{aligned}$$

~~Comparing~~ Comparing (*) and (***) shows $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$

2. ~~dm~~ Let $\gamma(t) = (\gamma_1, \gamma_2, \gamma_3)$
 $\mu(t) = (\mu_1, \mu_2, \mu_3)$

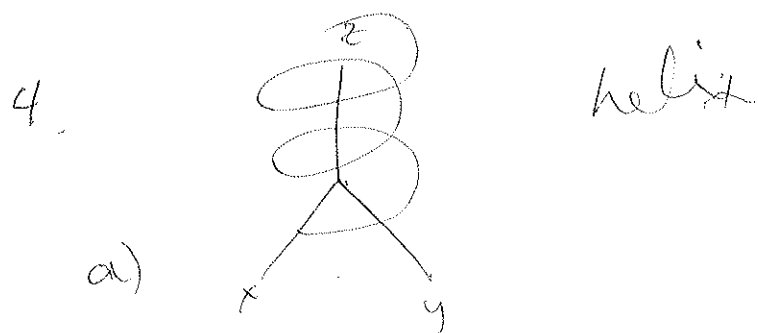
Then $\frac{d}{dt}(\gamma \cdot \mu) = \frac{d}{dt}(\gamma_1 \mu_1 + \gamma_2 \mu_2 + \gamma_3 \mu_3)$
 $= \dot{\gamma}_1 \mu_1 + \gamma_1 \dot{\mu}_1 + \dot{\gamma}_2 \mu_2 + \gamma_2 \dot{\mu}_2 + \dot{\gamma}_3 \mu_3 + \gamma_3 \dot{\mu}_3$
 $= \dot{\gamma}_1 \mu_1 + \dot{\gamma}_2 \mu_2 + \dot{\gamma}_3 \mu_3 + \gamma_1 \dot{\mu}_1 + \gamma_2 \dot{\mu}_2 + \gamma_3 \dot{\mu}_3$
 $= \dot{\gamma} \cdot \mu + \gamma \cdot \dot{\mu}$

$\frac{d}{dt}(\gamma \times \mu) = \frac{d}{dt}(\gamma_2 \mu_3 - \gamma_3 \mu_2, \gamma_3 \mu_1 - \gamma_1 \mu_3, \gamma_1 \mu_2 - \gamma_2 \mu_1)$
 $= (\dot{\gamma}_2 \mu_3 + \gamma_2 \dot{\mu}_3 - \dot{\gamma}_3 \mu_2 - \gamma_3 \dot{\mu}_2,$
 $\dot{\gamma}_3 \mu_1 + \gamma_3 \dot{\mu}_1 - \dot{\gamma}_1 \mu_3 - \gamma_1 \dot{\mu}_3,$
 $\dot{\gamma}_1 \mu_2 + \gamma_1 \dot{\mu}_2 - \dot{\gamma}_2 \mu_1 - \gamma_2 \dot{\mu}_1)$
 $= (\dot{\gamma}_2 \mu_3 - \dot{\gamma}_3 \mu_2, \dot{\gamma}_3 \mu_1 - \dot{\gamma}_1 \mu_3, \dot{\gamma}_1 \mu_2 - \dot{\gamma}_2 \mu_1)$
 $+ (\gamma_2 \dot{\mu}_3 - \gamma_3 \dot{\mu}_2, \gamma_3 \dot{\mu}_1 - \gamma_1 \dot{\mu}_3, \gamma_1 \dot{\mu}_2 - \gamma_2 \dot{\mu}_1)$
 $= \dot{\gamma} \times \mu + \gamma \times \dot{\mu}$

$$3. \quad \frac{d}{dt} (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) = \frac{d}{dt} (\dot{\mathbf{r}}) \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \frac{d}{dt} (\ddot{\mathbf{r}}) \\ = \ddot{\mathbf{r}} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \ddot{\mathbf{r}}'''$$

$\ddot{\mathbf{r}} \times \ddot{\mathbf{r}} = 0$ because $\mathbf{v} \times \mathbf{v} = 0$ for any vector $\mathbf{v}$.

$$\text{so } \frac{d}{dt} (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) = \dot{\mathbf{r}} \times \ddot{\mathbf{r}}'''$$



$$\dot{\mathbf{r}} = (-\sin t, \cos t, 1)$$

$$\|\dot{\mathbf{r}}\|^2 = (-\sin t)^2 + (\cos t)^2 + 1 = 2$$

$$\|\dot{\mathbf{r}}\| = \sqrt{2}$$

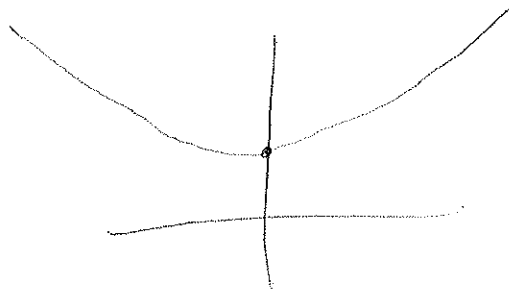
$$s(t) = \int_0^t \sqrt{2} \, du = \sqrt{2} \, t$$

$$b) \quad t = \frac{s}{\sqrt{2}}$$

unit speed reparametrization is

$$\tilde{\mathbf{r}}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right)$$

5. a)



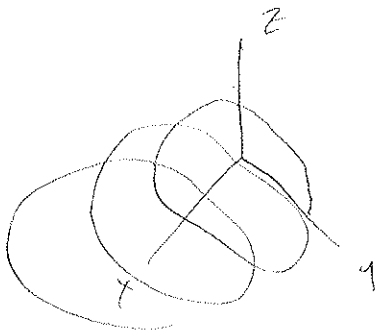
~~$\|\dot{\gamma}(t)\|^2$~~

$$\dot{\gamma}(t) = (1, \sinh(t), 0)$$

$$\|\dot{\gamma}(t)\|^2 = 1^2 + \sinh^2(t) > 0 \text{ for all } t$$

therefore $\gamma(t)$ is regular

b)



helix along x-axis

$$\dot{\gamma} = (2t, -2t \sin(2t), 2t \cos(2t))$$

$$= 2t (1, -\sin(2t), \cos(2t))$$

$$\dot{\gamma}(0) = 0$$

therefore γ is not regular.

$$6. \quad \gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, t \right)$$

$$\dot{\gamma}(t) = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, 1 \right)$$

$$\begin{aligned} \|\dot{\gamma}(t)\|^2 &= \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2} \\ &= 1 \end{aligned}$$

So γ is a unit-speed curve.